CAPTURING THE SYMMETRY OF ATTRACTORS 
AND THE TRANSITION TO SYMMETRIC 
CHAOS IN A VIBRO-IMPACT SYSTEM

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A three-degree-of-freedom vibro-impact system with symmetric two-sided constraints is considered. The system is strongly nonlinear and symmetric. The symmetric fixed point of the Poincaré map is deduced analytically, and the existence conditions of the symmetric fixed point are obtained. The six-dimensional Poincaré map can be expressed as the second iteration of another unsymmetric implicit map, which implies the symmetry of the Poincaré map. When the control parameter changes successively, symmetry-breaking bifurcation and symmetry-restoring bifurcation will occur at some point, and the attractor may change between symmetry and antisymmetry repeatedly. When a symmetry breaking bifurcation occurs, the symmetry is still the intrinsic property of the vibro-impact system. Here the Poincaré map cannot reflect the symmetry itself. However, the unsymmetric implicit map can capture a pair of antisymmetric ω-limit sets, which reflects the symmetry of the vibro-impact system. Different Poincaré sections are locally conjugate about a diffeomorphism. Therefore, as long as the perturbation is sufficiently small, changing the Poincaré section does not have any effect on the dynamical behavior. The transition to symmetric chaos is represented by numerical simulations.

Keywords: Vibro-impact; symmetry; attractor; Poincaré section; transition to symmetric chaos.

1. Introduction

Research into the dynamic behavior of vibro-impact systems has important significance in the optimization design of machinery and noise suppression. Hence, the complexity of the dynamics of vibro-impact system has received great attention. For a small ball bouncing vertically on a massive sinusoidally vibrating plate, it was shown that this deterministic dynamical system exhibits large families of irregular nonperiodic solutions in addition to the expected harmonic and subharmonic motions [Holmes, 1982]. An impacting oscillator that rebounds elastically was considered, and it was shown that it exhibits a family of subharmonic
resonant peaks while leading to chaos via cascades of period-doubling bifurcations [Thompson & Ghafari, 1982]. A periodically forced linear oscillator with impacts was considered, the dynamics was represented by a discontinuous map defined on the circle, and it was shown that the map undergoes period-doubling bifurcations followed by complex sequences of transition [Shaw & Holmes, 1983]. The effects of mass ratio, coefficient of restitution and gap size on the free and forced vibrations of an impact damper were studied [Bapat & Sankar, 1985]. The global dynamics and singularities in vibro-impact system were investigated [Whiston, 1992]. The performance and design of an impact damper were discussed for controlling the high amplitude vibration of a nonlinear oscillator [Chatterjee et al., 1995]. The motion of a single-degree-of-freedom forced oscillator subjected to a rigid amplitude constraint was considered, and the singularities caused by grazing impact were studied using analytical methods [Nordmark, 1991]. The rising phenomena which occur in sticking solutions of a two-degree-of-freedom impact oscillator was investigated [Wagg, 2004]. For the LR model motion in a horizontal impact oscillator, the stability, saddle-node and period-doubling bifurcation conditions were studied numerically and analytically [Luo, 2004]. The Poincaré map was established for vibro-impact systems, and the center manifold theorem technique and the theory of normal forms were applied to the systems, and various codimension two bifurcations, including Hopf-flip bifurcation, Hopf-Hopf bifurcation, and bifurcations of strong resonance cases were investigated in detail [Luo, 2006; Xie & Ding, 2005; Wen, 2001; Ding & Xie, 2005, 2006; Luo & Xie, 2001] for systems with impact, several methods for the calculation of Lyapunov exponents have been proposed [Stefanski, 2000; de Souza & Caldas, 2004; Jin et al., 2006; Awrejcewicz & Kudra, 2005]. In dynamical systems with symmetry, the bifurcations of attractors exhibit some nongeneric features, and the common occurrence of symmetry in both experimental design and theoretical models has motivated many researchers to develop many new theories. A single-degree-of-freedom impact oscillator having two-sided amplitude constraints was considered. The digital simulations indicated that when the stable and unstable manifolds intersect, a snale horseshoe exists [Shaw, 1985]. An inverted pendulum with symmetric rigid barriers was considered, and the global bifurcation which results in the creation of horseshoes was studied [Shaw & Rand, 1989]. Symmetry and bifurcations of a two-degree-of-freedom vibro-impact system with symmetric two-sided rigid constraints were studied [Yue & Xie, 2008]. The symmetry of Poincaré map was expanded to a three-degree-of-freedom vibro-impact system, and the effect of the symmetry of Poincaré map on possible bifurcations was represented in detail [Yue et al., 2009]. Based on the QR method, an unsymmetric implicit map was used to calculate all the Lyapunov exponents in a symmetric vibro-impact system [Yue & Xie, 2009]. In this paper, we consider the three-degree-of-freedom symmetric vibro-impact system. The symmetric fixed point of the Poincaré map is deduced analytically, and the existence conditions of the symmetric fixed point are obtained. The six-dimensional Poincaré map can be expressed as the second iteration of another unsymmetric implicit map, which implies the symmetry of the Poincaré map. It is shown that the unsymmetric implicit map captures the symmetry of the dynamical behavior. Different Poincaré sections are locally conjugate about a diffeomorphism. Therefore, if the vibro-impact system exhibits no grazing bifurcation, the vibro-impact system owns the same orbits and the same stability type for different Poincaré sections. That is, as long as the perturbation is sufficiently small, changing the Poincaré section does not have any dynamical effect. The transition to symmetric chaos is represented by numerical simulations.

2. Mechanical Model and the Symmetry of the Poincaré Map

A three-degree-of-freedom system subjected to periodic excitation is shown in Fig. 1. The system has three masses \( M_1, M_2, M_3 \). \( M_1 \) is connected to rigid plane via linear spring \( K_2 \) and linear viscous dashpot \( C_2 \). \( M_2 \) and \( M_3 \) are connected to \( M_2 \) via linear springs \( K_1 \) and \( K_3 \), and linear viscous dashpots \( C_1 \) and \( C_3 \), respectively. The excitation on mass \( M_i \) (\( i = 1, 2, 3 \)) is harmonic with amplitude \( P_i \). For small forcing amplitudes the system undergoes simple oscillations and behaves as a linear system. However, as the amplitudes are increased, \( M_2 \) begins to collide with two stops of \( M_1 \), and the system becomes discontinuous and strongly nonlinear. The impact is described by a coefficient of restitution \( R \). It is assumed that the duration of
and the nondimensional variables and parameters where the nondimensional differential equation of motion are given by

$$
U_{in} \ddot{x} + 2iU_{in} \ddot{u} + U_{ix} = U_{f} \sin(\omega t + \tau) \tag{1}
$$

where

$$
x = [x_1, x_2, x_3]^T,
U_{in} = \text{diag}[u_{m1}, u_{m2}, u_{m3}],
U_f = \text{diag}[u_{p1}, u_{p2}, u_{p3}],
U_i = \begin{bmatrix}
u_{c1} & -\nu_{c2} & 0 \\
-\nu_{c1} & \nu_{c1} + \nu_{c2} + \nu_{c3} & -\nu_{c3} \\
0 & -\nu_{c3} & \nu_{c3} \\
\end{bmatrix},
U_k = \begin{bmatrix}
u_{k1} & -\nu_{k1} & 0 \\
-\nu_{k1} & \nu_{k1} + \nu_{k3} & -\nu_{k3} \\
0 & -\nu_{k3} & \nu_{k3} \\
\end{bmatrix},
$$

and the nondimensional variables and parameters are

$$
t = T \sqrt{\frac{K_3}{M_3}} \xi = \frac{C_3}{2 \sqrt{K_3 M_3}} \omega = \sqrt{\frac{M_3}{K_3}}
$$

$$
f = \frac{P_3}{P_0}, \ u_{m1} = \frac{M_1}{M_3}, \ u_{k1} = \frac{K_1}{K_3},
$$

$$
u_{c1} = \frac{C_1}{C_3}, \ \nu_{k1} = \frac{P_1}{P_0}, \ x_1 = \frac{X_1}{K_3},
$$

$$
i = 1, 2, 3, \text{ and } P_0 = |P_1| + |P_2| + |P_3|.
$$

After each impact, the velocities of $M_2$ and $M_3$ change according to the impact law and the momentum conservation rule:

$$
y_{2+} = \delta_{11} y_{2-} + \delta_{12} y_{1-},
$$

$$
y_{3+} = \delta_{21} y_{2-} + \delta_{22} y_{3-},
$$

where $\delta_{11} = \frac{\nu_{c1} - \nu_{k1}}{\nu_{c1} + \nu_{c2} + \nu_{c3} - \nu_{k3}}, \ \delta_{12} = \frac{\nu_{c2} + \nu_{c3}}{\nu_{c1} + \nu_{c2} + \nu_{c3} - \nu_{k3}}, \ \delta_{21} = \frac{\nu_{k1}}{\nu_{c1} + \nu_{c2} + \nu_{c3} - \nu_{k3}}, \ \text{and} \ y_{2-} = \dot{x}_{2-} \text{ and} \ y_{3-} = \dot{x}_{3-} \text{ denote the nondimensional velocities of } M_i \text{ before and after impact, respectively.}

Let $x_1$ and $y_i$ denote the displacement and the velocity of $M_i$, respectively. Then between any two consecutive impacts, the nondimensional differential equation of motion can be rewritten as $\dot{X} = F(X, t)$ where $X = (x_1, y_1, x_2, y_2, x_3, y_3)^T$. The differential equation of motion satisfies both $F(-X, t + \frac{n\pi}{\omega}) = -F(X, t)$ ($n$ is an odd number) and $F(X, t + \frac{2n\pi}{\omega}) = F(X, t)$ [Yue & Xie, 2009].

The phase space of the vibro-impact system is $\mathbb{R}^6 \times S^1 = \{x_1, x_2, x_3, y_1, y_2, y_3 \mid (x_1, y_1, x_2, y_2, x_3, y_3) \in \mathbb{R}^6, t \in S^1\}$, (3)

where $S^1$ is the $2\pi$-circle. The Poincaré section $\Pi_0$ is chosen at the moment after the impact on the left side, and is expressed as

$$
\Pi_0 = \{(x_1, y_1, x_2, y_2, x_3, y_3, t) \in \mathbb{R}^6 \times S^1 \mid x_2 - x_3 = h, y_i = \dot{x}_i (i = 2, 3)\}, \tag{4}
$$

where $h = \frac{K_3 \dot{x}_3}{\omega}$. Then when the impact occurs on the right side, we define another section $\Pi_1$, which is expressed as

$$
\Pi_1 = \{(x_1, y_1, x_2, y_2, x_3, y_3, t) \in \mathbb{R}^6 \times S^1 \mid x_2 - x_3 = -h, y_i = \dot{x}_i (i = 2, 3)\}. \tag{5}
$$

And a transformation $R$ is defined as

$$
R : (x_1, y_1, x_2, y_2, x_3, y_3, t) \mapsto (-x_1, -y_1, -x_2 - y_2, -x_3 - y_3, t + \frac{n\pi}{\omega}). \tag{6}
$$

Defining $Q_1 : \Pi_0 \rightarrow \Pi_1$ (including an impact) and $Q_2(X_0) = X_0$, and defining $Q_2 : \Pi_1 \rightarrow \Pi_0$ (including an impact) and $Q_2(X_1) = X_1$, where $X_0, X_1 \in \Pi_0$ and $X_0, X_1 \in \Pi_1$, then the symmetry of the Poincaré map is expressed as $R \circ Q_1 = Q_2 \circ R$, indicating that $Q_1$ and $Q_2$ commute about $R$. Introducing a map $Q_0 = R^{-1} \circ Q_1$, we obtain the...
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Poincaré map as below [Yue & Xie, 2009]:

\[ P = Q_3 \circ Q_1 = Q_3^2, \quad P : \Omega_0 \rightarrow \Omega_0, \]  

(7)

where the symbol “\( \circ \)” denotes the composition of two maps. Equation (7) shows that the Poincaré map \( P \) is the second iteration of \( Q_3 \), where \( Q_3 \) has no symmetry.

3. Symmetric Fixed Point and Antisymmetric Fixed Point

If \( X_0 \in \Omega_0 \) satisfies \( P(X_0) = X_0 \), then \( X_0 \) is a fixed point of the Poincaré map \( P \), corresponding to the associated periodic motion of the system. Moreover, if the fixed point \( X_0 \) satisfies \( Q_3(X_0) = X_0 \), then \( X_0 \) is a symmetric fixed point (or symmetric period \( n \)-fold fixed point) of the Poincaré map \( P \), corresponding to the associated symmetric period \( n \)-fold motion of the system.

The eigenfrequencies of Eq. (1) can be solved as \( \omega_1, \omega_2, \omega_3 \). Taking \( \Psi \) as the canonical model matrix, and making the change of variable \( [x_1, x_2, x_3]^T = \Psi \eta \), Eq. (1) becomes

\[ i \xi + C_\xi + A_\xi = \Psi^T U_{1 f} \sin(\omega t + \tau), \]

(8)

where \( A = \text{diag} [\omega_1^2, \omega_2^2, \omega_3^2] \), \( C = 2 \omega A \).

Let \( \phi_{ij} \) denote the element of \( \Psi \), the general solution of Eq. (1) is given by

\[ x_i(t) = \sum_{j=1}^{3} \phi_{ij} \left( e^{-\eta_j t}(a_j \cos(\omega_j t) + b_j \sin(\omega_j t)) \right) + A_j \sin(\omega t + \tau) + B_j \cos(\omega t + \tau), \]

(9)

where \( \eta_j = \sqrt{\omega_j^2 - \delta_j^2} \), \( \omega_j = \sqrt{\omega_j^2 - \delta_j^2} \), and \( a_j \) and \( b_j \) are integration constants, \( A_j \) and \( B_j \) are amplitude constants.

Let the origin of the time coordinate be displaced to the moment that \( M_3 \) impacts on the left of \( M_2 \) (\( t = t_0 = 0 \)). Subsequently, at the moment \( t = t_1 = \frac{\pi}{\omega_3} \), \( M_3 \) impacts on the right. At the moment \( t = t_2 = \frac{\pi}{\omega_3} \), \( M_3 \) impacts on the left once again.

Proposition 1 (Existence of the symmetric fixed point). If there are initial conditions \( \tau = \tau_0, x_j(0) = x_{0j}, x_{1+} = x_{0+}, \) which makes the solution \( x_i(t) \) in the interval \( t \in [0, \frac{\pi}{\omega_3}] \) satisfy

\[ x_i \left( \frac{n\pi}{\omega_3} \right) = -x_i(0), \quad x_{1+} \left( \frac{n\pi}{\omega_3} \right) = -x_{1+}(0), \]

(10-1)

\[ x_2(0) - x_3(0) = h, \quad x_2 \left( \frac{n\pi}{\omega_3} \right) - x_3 \left( \frac{n\pi}{\omega_3} \right) = -h, \]

(10-2)

\[ |x_2(t) - x_3(t)| \leq h, \quad t \in \left( 0, \frac{n\pi}{\omega_3} \right), \]

(10-3)

then the symmetric fixed point exists.

Substituting Eq. (9) into Eq. (10), \( \tau_0, a_j, b_j \) can be solved. Then substituting \( t = 0 \) and \( \tau_0, a_j, b_j \) into Eq. (9), we can obtain the symmetric fixed point \( X_0 \).

If \( X_0 \) is a fixed point of the Poincaré map, and \( Q_3(X_0) = X_0 \neq X_0 \), we have

\[ P(X_0) = Q_3^2(X_0) = Q_3 \circ Q_3(X_0) = Q_3 \circ P(X_0) = Q_3 \circ Q_3(X_0) = X_0, \]

(11)

hence \( X_0 \) is the fixed point of the Poincaré map.

In addition,

\[ Q_3(X_0) = Q_3^2(X_0) = P(X_0) = X_0, \]

(12)

\( X_0 \), and \( X_3 \) are a pair of antisymmetric fixed points, which correspond to a pair of antisymmetric period \( n \)-fold motions.

The eigenvalues of the Jacobi matrix \( DP(X_0) \) determine the stability of the symmetric fixed point \( X_0 \) of the Poincaré map. Suppose that all the eigenvalues of \( DP(X_0) \) lie inside the unit circle, the symmetric fixed point \( X_0 \) is stable. If there are some eigenvalues crossing the unit circle, various bifurcations take place. For example, when there is a real eigenvalue crossing the unit circle at \( r + 1 \), the symmetric fixed point changes its stability, and bifurcates into a pair of antisymmetric fixed points via pitchfork bifurcation. If there is a conjugate pair of complex eigenvalues crossing the unit circle, the Neimark-Sacker bifurcation of symmetric fixed point takes place. However, for the symmetric fixed point, the symmetry of the Poincaré map suppresses codimension-1 period-doubling bifurcation, Hopf-flip bifurcation and pitchfork-flip bifurcation completely [Yue et al., 2009].

4. Capturing the Symmetry of Attractors by the Map \( Q_3 \)

In vibro-impact systems, when the grazing phenomenon occurs, the Poincaré map becomes singular. In this paper, the singularity induced by
the discontinuity of the Poincaré map is not considered. Hence, it is assumed that \( Q_1, Q_2, Q_3, P \) are all continuous and invertible.

**Definition 2** [Robinson, 1995] (\( \omega \)-limit set). Let \( B \) be a complete metric space with metric \( d \), and \( h : B \rightarrow B \) a continuous map. A point \( Y \) is an \( \omega \)-limit point of \( X \) for \( h \) provided there exists a sequence of \( n_k \) going to infinity as \( k \) goes to infinity such that \( \lim_{k \rightarrow \infty} h^n(X) = Y \). The set of all \( \omega \)-limit points of \( X \) for \( h \) is called the \( \omega \)-limit set of \( X \), and is denoted by \( \omega(X) \).

The \( \omega \)-limit sets of \( X \) generated by the iterations of the \( P \) map and the \( Q_1 \) map are denoted by \( \omega(P)(X) \) and \( \omega(Q_1)(X) \), respectively. A limit set can be attracting or nonattracting. In this paper, we define a point \( X \) to be an asymptotically stable \( \omega \)-limit set if \( \omega(P)(X) = \omega(Q_1)(X) \) (i.e., \( P \) and \( Q_1 \) have the same \( \omega \)-limit set). Besides, according to Eq. (17), if \( \omega(P)(X) \) is symmetric, if it is mapped onto itself under the \( Q_1 \) map, then \( \omega(P)(X) \) is symmetric.

As a physical parameter changes, a symmetry breaking bifurcation occurs when a symmetric limit set becomes nonsymmetric, and a symmetry restoring bifurcation occurs when a nonsymmetric limit set becomes symmetric. Since the \( \omega \)-limit set may change between symmetry and antisymmetry repeatedly, the symmetric vibro-impact system could alternate between symmetry breaking and symmetry restoring many times before entering into chaos. When a symmetry-breaking bifurcation occurs, the symmetry is still the intrinsic property of the vibro-impact system. Here the Poincaré map \( P \) cannot reflect the symmetry itself. However, Eq. (15) implies that the unsymmetric implicit map \( Q_1 \) can capture a pair of antisymmetric \( \omega \)-limit sets, which reflects the symmetry of the vibro-impact system.

5. Varying the Poincaré Section

Consider two \( C^0 \) diffeomorphisms \( f : \mathbb{R}^n \rightarrow \mathbb{R}^n \) and \( g : \mathbb{R}^n \rightarrow \mathbb{R}^n \). If \( f \) and \( g \) are \( C^0 \) conjugate (\( k \leq \tau \)) if there exists a \( C^0 \) diffeomorphism \( h : \mathbb{R}^n \rightarrow \mathbb{R}^n \) such that \( g = h \circ f \). If \( k = 0 \), \( f \) and \( g \) are topologically conjugate. According to Wiggins [1990], if \( f \) and \( g \) are \( C^0 \) conjugate, then orbits of \( f \) map orbits of \( g \) under \( h \). Moreover, If \( f \) and \( g \) are \( C^0 \) conjugate, \( k \geq 1 \), and \( x_0 \) is a fixed point of \( f \), then the eigenvalues of \( Df(x_0) \) are equal to the eigenvalues of \( Dg(b(x_0)) \). Here \( b \) need not be defined on all of \( \mathbb{R}^n \) but possibly only locally about a given point.

As shown in Fig. 3, for the vibro-impact system we considered, let \( X_0^3 \) and \( X_1^3 \) be two points on the periodic solution, and let \( \Pi_0 \) and \( \Pi_1 \) be two \( (n - 1) \)-dimensional cross-sections at \( X_0^3 \) and \( X_1^3 \), respectively, which are transverse to the vector field. \( h_2 \) and \( h_4 \) represent the map from the section before the impact to the section after the impact, and \( h_1, h_3 \) represent the map of the flow generated by Eq. (1). If the grazing phenomenon is not considered, there exists two \( C^0 \) diffeomorphisms

\[
h_4 : \Pi_0 \rightarrow \Pi_1, \quad h = h_2 \circ h_1
\]
Then the two Poincaré maps can be defined as
\[ P_0 : V_0 \to V_0 ; \quad X^*_0 \mapsto h_b \circ h_a (X^*_0), \]
\[ X^*_0 \in V_0 \subset \Pi_0, \] (20)
\[ P_1 : V_1 \to V_1 ; \quad X^*_1 \mapsto h_a \circ h_b (X^*_1), \]
\[ X^*_1 \in V_1 \subset \Pi_1. \] (21)

Proposition 3. \( P_0 \) and \( P_1 \) are locally conjugate if the vibro-impact system exhibits no grazing bifurcation.

Proof. Because the vibro-impact system exhibits no grazing bifurcation, \( h_a \) and \( h_b \) are two \( C^r \) diffeomorphisms. Since \( h_a(\Pi_0) = \Pi_1, P_0(V_0) \subset \Pi_0, P_1(V_1) \subset \Pi_1 \), and according to Eqs. (20) and (21), we have \( P_1 \circ h_a = h_a \circ P_0 \), then \( P_0 \) and \( P_1 \) are locally conjugate.

Therefore, if the vibro-impact system exhibits no grazing bifurcation, the vibro-impact system owns the same orbits and the same stability type for different Poincaré sections. As long as the perturbation is sufficiently small, changing the Poincaré section does not have any dynamical effect. ■

6. Numerical Analysis

Now system with the second set of parameters (2):
\[ \zeta = 0.008, \ \rho = 0.8, \ \alpha = 0.04, \ u_{m1} = 0.45, \ u_{m2} = 2.5, \ u_{n1} = 1, \ u_{k1} = 0.8, \ u_{k2} = 0.5, \ u_{k3} = 1, \ u_{f1} = 0.5, \ u_{f2} = 1, \ u_{f3} = 1 \] is considered.

When the exciting frequency \( \omega \) is chosen as the control parameter, the bifurcation diagrams of the map \( Q \), are represented in Fig. 4. Figure 4(a) shows the bifurcation diagram of the even number of iterations under the map \( Q^2 \), and Fig. 4(b) shows the bifurcation diagram of the odd number of iterations under the map \( Q^3 \). When the control parameter \( \omega \) decreases, a symmetric fixed point bifurcates into two antisymmetric fixed points via pitchfork bifurcation firstly. Hence, the symmetry of the periodic attractor is broken at the pitchfork bifurcation point \( \omega_b = 2.9452 \). Subsequently, the two antisymmetric fixed points bifurcate into two antisymmetric invariant circles, and generates a pair of antisymmetric bifurcation sequences as the parameters decrease continuously. However, when the control parameter \( \omega \) decreases to \( \omega_r = 2.7988 \), the two antisymmetric attractors intersect with
each other and merge leading to a single symmetric one, and the symmetry of the attractor is restored at this point. Certainly, the attractor may change amidst long periodicity, quasi-periodicity and chaos repeatedly. Moreover, with \( \omega < \omega_r \), the attractor may change between symmetry and antisymmetry repeatedly.

With \( \omega = 2.93 \), the pitchfork bifurcation occurs, and there exists three periodic motions. Figure 5 represents one unstable symmetric periodic motion and two stable antisymmetric periodic motions. The vertical boldfaced line represents the impact behavior. \( A \to B \to C \to D \) represents the phase trajectory of one antisymmetric periodic motion, and \( A' \to B' \to C' \to D' \) represents the phase trajectory of another antisymmetric periodic motion. \( A \) and \( C' \) are symmetric about the origin, so is \( A' \) and \( C \). \( A \to B \to C \) (or \( A' \to B' \to C' \)) denotes the map \( Q_1 \), \( C \to D \to A \) (or \( C' \to D' \to A' \)) denotes the map \( Q_2 \).

Figure 6 represents the phase trajectories corresponding to the map \( Q_1 \) on the phase plane \((x_2, y_2)\). In Fig. 6(a), \( A \to B \to C \) and \( C \to A \) denotes the map \( Q_1 \) and the transformation \( R^{-1} \).

\[ fig. 5 \text{ The phase trajectories corresponding to the Poincaré map } P, \text{ the symmetric periodic motion and the two antisymmetric periodic motions: (a) The phase plane } (x_2, y_2); \text{ (b) the phase plane } (x_3, y_3). \]

\[ fig. 6 \text{ The phase trajectories corresponding to the map } Q_1 \text{ on the phase plane } (x_2, y_2): \text{ (a) The symmetric periodic motion, (b) capturing the two antisymmetric periodic motions.} \]
respectively. Then $A \rightarrow B \rightarrow C \rightarrow A$ implies $Q_1(A) = R^{-1}Q_1(A) = A$, indicating that $A$ is a symmetric fixed point corresponding to the symmetric periodic motion. In Fig. 6(b), $A$ and $A'$ denote the initial point of the two antisymmetric periodic motions (i.e. after impact occurs on the left), corresponding to the fixed point of the Poincaré map $P$. Since $R$ is a symmetric transformation, then $R^{-1}$ is also a symmetric one. The map $Q_1(A) = R^{-1}Q_1(A) = A'$ implies that the phase trajectory setting out from $A$ undergoes the map $Q_1$ firstly, and reaches the point $A'$ via transformation $R^{-1}$ subsequently, and the map $Q_3$ is omitted. Therefore, the Poincaré map $P = Q_{2\gamma}$ represents the phase trajectory $A \rightarrow B \rightarrow C \rightarrow A' \rightarrow B' \rightarrow C'$, indicating that the iteration of the map $Q_\gamma$ can capture the two antisymmetric fixed points $A$ and $A'$. It should be noted that the map $Q_\gamma$ is a virtual implicit map, and cannot represent the real map of the vibro-impact system. Therefore, the motion corresponding to the map $Q_\gamma$ is a virtual motion.

When $\omega$ is decreasing, the attractors of the $P$ map and the $Q_\gamma$ map, represented by the projected Poincaré section, are shown in Fig. 7. The transition to symmetric chaos is as follows: one symmetric fixed point $\rightarrow$ a pair of antisymmetric fixed points $\rightarrow$ a pair of antisymmetric quasi-periodic attractors [Fig. 7(a)] $\rightarrow$ a pair of antisymmetric

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**Fig. 7.** Transition to symmetric chaos: (a) $\omega = 2.81$, (b) $\omega = 2.8$, (c) $\omega = 2.795$, (d) $\omega = 2.793$, (e) $\omega = 2.785$, (f) $\omega = 2.78$, (g) $\omega = 2.77$, (h) $\omega = 2.68$. 

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chaotic attractors [Fig. 7(b)] $\rightarrow$ one symmetric chaotic attractor [Fig. 7(c)] $\rightarrow$ one symmetric quasi-periodic attractor [Fig. 7(d)] $\rightarrow$ a pair of antisymmetric quasi-periodic attractors [Fig. 7(e)] $\rightarrow$ a pair of antisymmetric chaotic attractors [Fig. 7(f)] $\rightarrow$ one symmetric quasi-periodic attractor [Fig. 7(g)] $\rightarrow$ one symmetric chaotic attractor [Fig. 7(h)]. In Figs. 7(a), 7(b), 7(e) and 7(f), since $\omega_P(X) \neq \omega_Q(X)$, the attractors are all antisymmetric. The two-dimensional phase plane portrait is the projection of the six-dimensional phase space, and the two conjugate attractors shown in Figs. 7(b), 7(c) and 7(f) seem to have an intersection. In fact, the two conjugate attractors do not intersect in such three cases, and the attractors are all antisymmetric.

Figure 8 represents the bifurcation diagrams on two cross-sections $\Pi_0$ and $\Pi_1$ expressed by Eqs. (4) and (5), respectively. It is shown that the symmetry of the periodic attractor on two cross-sections is broken at the same point $\omega_b = 2.9452$. When the control parameter $\omega$ decreases to $\omega_r = 2.7988$, the two antisymmetric attractors on two cross-sections restore the symmetry at the same time. Varying the control parameter $\omega$ successively, the vibro-impact system exhibits two synchronous bifurcation sequences. This is because different Poincaré sections $\Pi_0$ and $\Pi_1$ are locally conjugate about the diffeomorphisms $Q_1$. Therefore, as long as the perturbation is sufficiently small, changing the Poincaré section does not have any effect on the dynamical behavior.
7. Conclusions

For the three-degree-of-freedom symmetric vibro-impact system, the six-dimensional Poincaré map can be expressed as the second iteration of another unsymmetric implicit map, which implies the symmetry of the Poincaré map. When the control parameter changes successively, symmetry-breaking bifurcation and symmetry-restoring bifurcation will occur at some point, and the attractor may change between symmetry and antisymmetry repeatedly. It is shown that the unsymmetric implicit map captures the symmetry of the dynamics.

Different Poincaré sections are locally conjugate about a diffeomorphism. Therefore, if the vibro-impact system exhibits no grazing bifurcation, the vibro-impact system owns the same orbits and the same stability type for different Poincaré sections. That is, as long as the perturbation is sufficiently small, changing the Poincaré section does not have any effect on the dynamical behavior. The typical route to chaos is also represented by numerical analysis. When the control parameter varies continuously, a symmetric fixed point bifurcates into two antisymmetric fixed points via pitchfork bifurcation at first, and the symmetry is broken at this point. Subsequently, the two antisymmetric fixed points bifurcate into two antisymmetric invariant circles, and generate a pair of antisymmetric bifurcation sequences. When the parameter changes to some point, two antisymmetric attractors will intersect with each other and merge leading to a single symmetric one, which indicates that symmetry is restored at this point.

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References


